Multiple Quantum Wells

Reading: Brennan - 2.5.

Lecture by Shamas M. Ummer

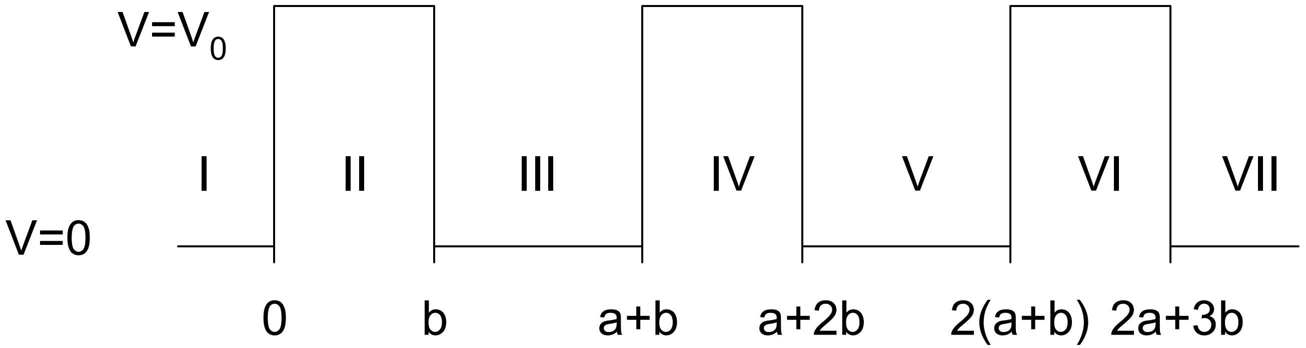
School of Electrical and Computer Engineering
Georgia Institute of Technology

Multiple Quantum Wells

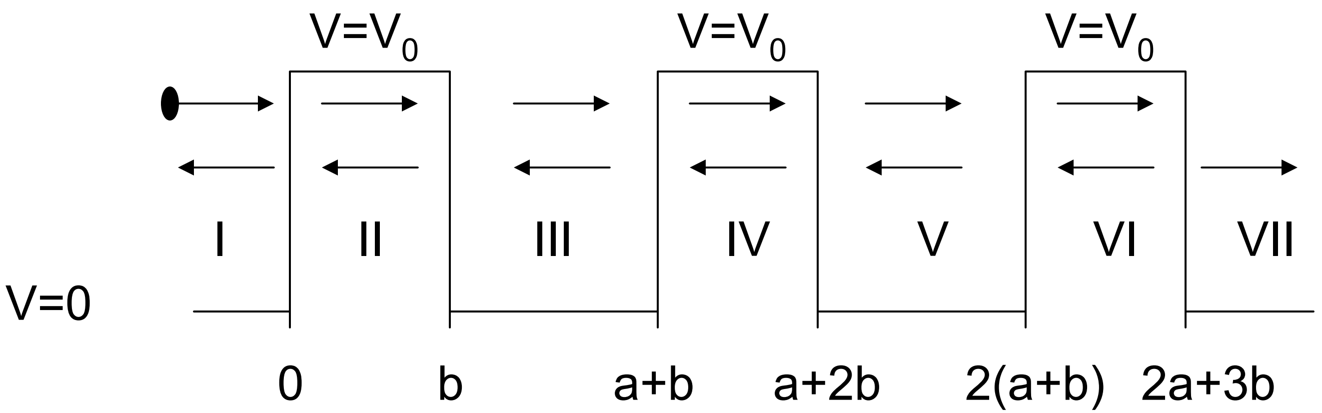
- This technology is used extensively in semiconductor laser technology
- Multiple Quantum Wells are grown using MBE, CBE and MOCVD
- Typical materials used:
  - GaAs/AlGaAs
  - InGaAs/InP
  - GaN/AlGaN

Multiple Quantum Wells – General Structure ($E<V_0$)

The general structure of a Multiple Quantum well with potential V_0, barrier width "b" and well width "a" is shown below:



Let us consider the case when an electron with energy less than V_0 is incident on the structure:



We already have the necessary tools to solve this problem. We take the same approach as we did for the potential well and potential barrier problem, i.e., we solve for the wave function in each region and then we apply the boundary conditions between the regions and finally, solve for the reflection and transmission probabilities.

Multiple Quantum Wells – Assumptions (for both cases)

Two important Assumptions are made for this problem:

– *Electron effective mass is assumed to be the same in all regions*

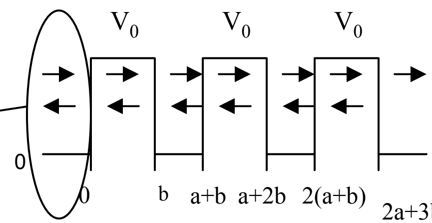
This allows us to use the simple boundary conditions we have been using for solving our barrier and well problems

– *No field is applied to the structure*

Therefore, the device is translationally symmetric, i.e., the electron energy doesn't change according to position.

Multiple Quantum Wells – Wave Functions Review ($E<V_0$)

Now, we can begin solving the problem. For all the regions, the wave function can be found by solving the Schrödinger Equation. For region I:



$$\frac{-\hbar^2}{2m}\frac{d^2\Psi_1}{dx^2}+\cancelto{0}{V_0}\Psi_1 = E\Psi_1$$

$$\Rightarrow \Psi_1 = \underbrace{e^{ik_1x}}_{\text{Incident wave}} + \underbrace{R\,e^{-ik_1x}}_{\text{Reflected wave}}$$

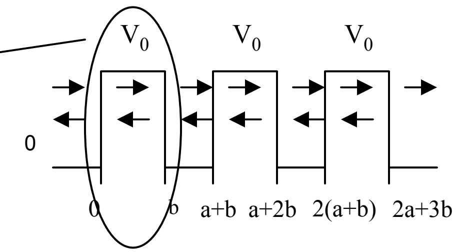
where,

$$k_1 = \sqrt{\frac{2mE}{\hbar^2}}$$

$$R = reflection\ co-efficient$$

Multiple Quantum Wells – Wave Functions Review ($E<V_0$)

For region II, we have a potential of V_0:



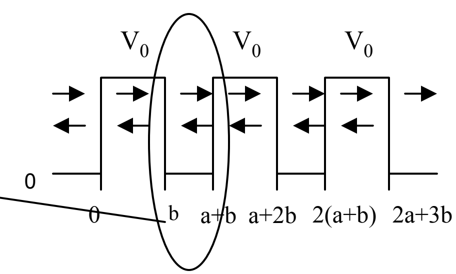
$$\frac{-\hbar^2}{2m}\frac{d^2\Psi_2}{dx^2}+V_0\Psi_2=E\Psi_2$$

$$\Rightarrow \Psi_2 = C_2^+ e^{k_2 x} + C_2^- e^{-k_2 x} \quad \text{where,} \quad k_2=\sqrt{\frac{2m(V_0-E)}{\hbar^2}}$$

+x direction wave **-x direction wave**

Or, in terms of hyperbolic functions, $\Psi_2 = C_2^+ \cosh k_2 x + C_2^- \sinh k_2 x$

For region III,



$$\frac{-\hbar^2}{2m}\frac{d^2\Psi_3}{dx^2}+\cancel{V_0}^{\,0}\Psi_3=E\Psi_3$$

$$\Psi_3 = C_3^+ e^{ik_1 x} + C_3^- e^{-ik_1 x} \quad \text{where,} \quad k_1=\sqrt{\frac{2mE}{\hbar^2}}$$

Or, $\Psi_3 = C_3^+ \cos k_1 x + C_3^- \sin k_1 x$

+x direction wave **-x direction wave**

Similarly, we can solve for the wave functions for the other regions with each region with a potential having similar solutions as Ψ_2 and the regions without a potential having solutions similar to Ψ_3

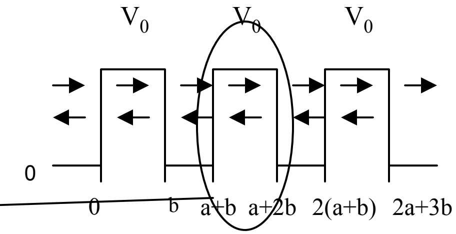
Multiple Quantum Wells – Wave Functions Review (E<V$_0$)

For Region IV:

$$\Psi_4 = C_4^+ e^{k_2 x} + C_4^- e^{-k_2 x},$$

$$k_2 = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$$

$$\Psi_4 = C_4^+ \cosh k_2 x + C_4^- \sinh k_2 x$$

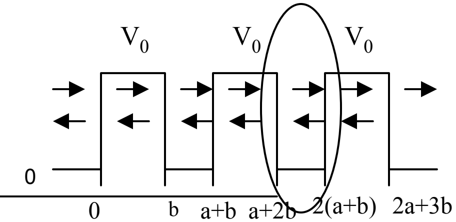


For Region V:

$$\Psi_5 = C_5^+ e^{ik_1 x} + C_5^- e^{-ik_1 x},$$

$$k_1 = \sqrt{\frac{2mE}{\hbar^2}}$$

$$\Psi_5 = C_5^+ \cos k_1 x + C_5^- \sin k_1 x$$

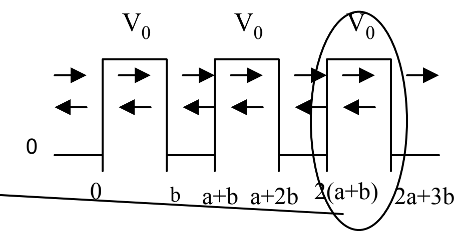


For Region VI:

$$\Psi_6 = C_6^+ e^{k_2 x} + C_6^- e^{-k_2 x},$$

$$k_2 = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$$

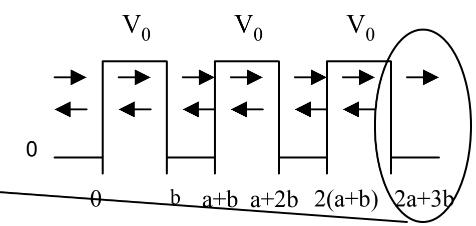
$$\Psi_6 = C_6^+ \cosh k_2 x + C_6^- \sinh k_2 x$$



For Region VII:

$$\Psi_7 = T e^{ik_1 x},$$

$$k_1 = \sqrt{\frac{2mE}{\hbar^2}}$$

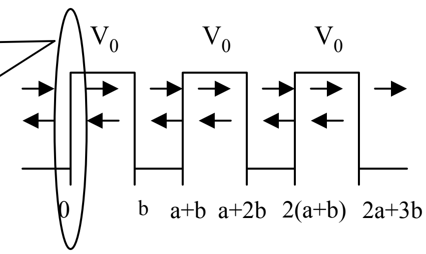


Note: For region VII, only a +x component exists as the wave extends out to infinity and there is no barrier to reflect the wave back

Multiple Quantum Wells – Boundary Conditions (I-II interface) ($E<V_0$)

Now, we can apply the boundary conditions between each region. For regions I and II, we can use two boundary conditions:

- *Boundary Condition 1 (Continuity of the wave function):*



$$\Psi_1(x=0) = \Psi_2(x=0)$$

$$\left(e^{ik_1x} + R\,e^{-ik_1x}\right)\Big|_{x=0} = \left(C_2^+ \cosh k_2x + C_2^- \sinh k_2x\right)\Big|_{x=0}$$

$$1 + R = C_2^+ \qquad (Eq.\,1)$$

- *Boundary Condition 2 (Probability Current Density):*

$$j_{1x}\big|_{x=0} = j_{2x}\big|_{x=0}$$

$$j_{1x} = \frac{\hbar}{2mi}\left(\Psi_1^*\,\vec{\nabla}\Psi_1 - \Psi_1\,\vec{\nabla}\Psi_1^*\right) = \left(\frac{ik_1}{m}e^{ik_1x} - \frac{ik_1}{m}R\,e^{-ik_1x}\right)$$

$$j_{2x} = \frac{\hbar}{2mi}\left(\Psi_2^*\,\vec{\nabla}\Psi_2 - \Psi_2\,\vec{\nabla}\Psi_2^*\right) = \left(\frac{k_2}{m}C_2^+ \sinh k_2x + \frac{k_2}{m}C_2^- \cosh k_2x\right)$$

$$\therefore, \left(\frac{ik_1}{m}e^{ik_1x} - \frac{ik_1}{m}R\,e^{-ik_1x}\right)\Bigg|_{x=0} = \left(\frac{k_2}{m}C_2^+ \sinh k_2x + \frac{k_2}{m}C_2^- \cosh k_2x\right)\Bigg|_{x=0}$$

$$\frac{ik_1}{m} - \frac{ik_1}{m}R = \frac{k_2}{m}C_2^- \qquad (Eq.\,2)$$

NOTE: For every interface, the boundary conditions are similar. The only parameter that changes is the x value at the interface.

Multiple Quantum Wells – Boundary Conditions (I-II interface) ($E<V_0$)

But, equation 1 and 2 can also be expressed in matrix form

$$1 + R = C_2^+ \qquad (Eq.\,1)$$

$$\frac{ik_1}{m} - \frac{ik_1}{m}R = \frac{k_2}{m}C_2^- \qquad (Eq.\,2)$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 1 \\ \frac{ik_1}{m} & \frac{-ik_1}{m} \end{bmatrix}\begin{bmatrix} 1 \\ R \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{k_2}{m} \end{bmatrix}\begin{bmatrix} C_2^+ \\ C_2^- \end{bmatrix}$$

$$Solving\ for \begin{bmatrix} 1 \\ R \end{bmatrix},$$

$$\begin{bmatrix} 1 \\ R \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \frac{ik_1}{m} & \frac{-ik_1}{m} \end{bmatrix}^{-1}\begin{bmatrix} 1 & 0 \\ 0 & \frac{k_2}{m} \end{bmatrix}\begin{bmatrix} C_2^+ \\ C_2^- \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ R \end{bmatrix} = \left(\frac{-m}{2ik_1}\right)\begin{bmatrix} \frac{-ik_1}{m} & -1 \\ \frac{-ik_1}{m} & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{k_2}{m} \end{bmatrix}\begin{bmatrix} C_2^+ \\ C_2^- \end{bmatrix}$$

MATRIX IDENTITIES USED HERE:

$1.\,if\ AB = C, then\ B = A^{-1}C$

$2.\,if\ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, then\ A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

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Multiple Quantum Wells – Boundary Conditions (I-II interface) (E<V$_0$)

(CONTD.)

$$\begin{bmatrix} 1 \\ R \end{bmatrix} = \left(\frac{-m}{2ik_1}\right) \begin{bmatrix} \frac{-ik_1}{m} & -1 \\ \frac{-ik_1}{m} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{k_2}{m} \end{bmatrix} \begin{bmatrix} C_2^+ \\ C_2^- \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ R \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{2} & \frac{m}{2ik_1} \\ \frac{1}{2} & \frac{-m}{2ik_1} \end{bmatrix}}_{[a]} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & \frac{k_2}{m} \end{bmatrix}}_{[b]} \begin{bmatrix} C_2^+ \\ C_2^- \end{bmatrix}$$

x

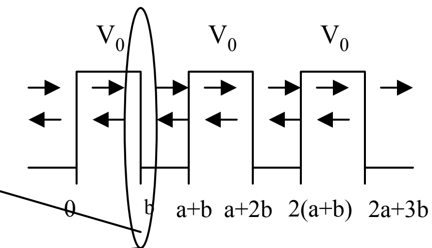
$$\begin{bmatrix} 1 \\ R \end{bmatrix} = \left[Y_{I-II}\right] \begin{bmatrix} C_2^+ \\ C_2^- \end{bmatrix} \qquad (Eq.\ 3)$$

where, Y_{I-II} is a 2x2 matrix that is obtained by multiplying [a] and [b]. Y_{I-II} is called the ***transfer matrix*** for the region I-II interface (A *transfer matrix* describes the boundary conditions at each interface).

Note: The subscripts of $Y_{\#1-\#2}$ stands for the regions #1-#2 interface it represents

Multiple Quantum Wells – Boundary Conditions (II-III interface) ($E<V_0$)

We can follow the same procedure we did for the regions I-II interface for obtaining the transfer matrix for the region II-III. Again, we start by using the boundary conditions @ x = b for regions II and III:



$$BC1: \Psi_2(x=b) = \Psi_3(x=b)$$

$$\therefore, C_2^+ \cosh k_2 b + C_2^- \sinh k_2 b = C_3^+ \cos k_1 b + C_3^- \sin k_1 b$$

$$BC2: j_{2x}\big|_{x=b} = j_{3x}\big|_{x=b}$$

$$j_{2x} = \frac{\hbar}{2mi}\left(\Psi_2^* \vec{\nabla}\Psi_2 - \Psi_2 \vec{\nabla}\Psi_2^*\right) = \frac{k_2}{m} C_2^+ \sinh k_2 x + \frac{k_2}{m} C_2^- \cosh k_2 x$$

$$j_{3x} = \frac{\hbar}{2mi}\left(\Psi_3^* \vec{\nabla}\Psi_3 - \Psi_3 \vec{\nabla}\Psi_3^*\right) = -\frac{k_1}{m} C_3^+ \sin k_1 x + \frac{k_1}{m} C_3^- \cos k_1 x$$

$$\therefore, \frac{k_2}{m} C_2^+ \sinh k_2 b + \frac{k_2}{m} C_2^- \cosh k_2 b = -\frac{k_1}{m} C_3^+ \sin k_1 b + \frac{k_1}{m} C_3^- \cos k_1 b$$

Again, we can represent these two equations in in the matrix form:

$$\begin{bmatrix} \cosh k_2 b & \sinh k_2 b \\ \frac{k_2}{m}\sinh k_2 b & \frac{k_2}{m}\cosh k_2 b \end{bmatrix} \begin{bmatrix} C_2^+ \\ C_2^- \end{bmatrix} = \begin{bmatrix} \cos k_1 b & \sin k_1 b \\ -\frac{k_1}{m}\sin k_1 b & \frac{k_1}{m}\cos k_1 b \end{bmatrix} \begin{bmatrix} C_3^+ \\ C_3^- \end{bmatrix}$$

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Multiple Quantum Wells – Boundary Conditions (II-III interface) ($E<V_0$)

(CONTD.)

$$\begin{bmatrix} \cosh k_2 b & \sinh k_2 b \\ \dfrac{k_2}{m}\sinh k_2 b & \dfrac{k_2}{m}\cosh k_2 b \end{bmatrix}\begin{bmatrix} C_2^+ \\ C_2^- \end{bmatrix} = \begin{bmatrix} \cos k_1 b & \sin k_1 b \\ -\dfrac{k_1}{m}\sin k_1 b & \dfrac{k_1}{m}\cos k_1 b \end{bmatrix}\begin{bmatrix} C_3^+ \\ C_3^- \end{bmatrix}$$

$$Solving \quad for \begin{bmatrix} C_2^+ \\ C_2^- \end{bmatrix},$$

$$\begin{bmatrix} C_2^+ \\ C_2^- \end{bmatrix} = \frac{m}{k_2}\begin{bmatrix} \dfrac{k_2}{m}\cosh k_2 b & -\sinh k_2 b \\ -\dfrac{k_2}{m}\sinh k_2 b & \cosh k_2 b \end{bmatrix}\begin{bmatrix} \cos k_1 b & \sin k_1 b \\ -\dfrac{k_1}{m}\sin k_1 b & \dfrac{k_1}{m}\cos k_1 b \end{bmatrix}\begin{bmatrix} C_3^+ \\ C_3^- \end{bmatrix}$$

x

$$\begin{bmatrix} C_2^+ \\ C_2^- \end{bmatrix} = \left[Y_{II-III}\right]\begin{bmatrix} C_3^+ \\ C_3^- \end{bmatrix}$$

$$Substituting\ the\ above\ in\ Eq.3, we\ have$$

$$\begin{bmatrix} 1 \\ R \end{bmatrix} = \left[Y_{I-II}\right]\begin{bmatrix} C_2^+ \\ C_2^- \end{bmatrix} = \left[Y_{I-II}\right]\left[Y_{II-III}\right]\begin{bmatrix} C_3^+ \\ C_3^- \end{bmatrix} \qquad (Eq.\ 4)$$

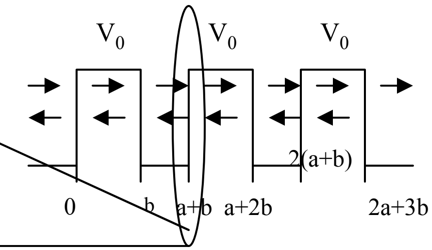
Note: Now, it should be obvious to see why we use the $Y_{\#1\text{-}\#2}$ notation - we avoid rewriting the rather nasty matrix multiplication with this notation.

Multiple Quantum Wells – Boundary Conditions (III-IV interface) (E<V_0)

We continue finding the transfer matrices. Using the boundary conditions @ x = a + b for regions III and IV:

$$BC1:\ \Psi_3(x=a+b)=\Psi_4(x=a+b)$$

$$\therefore,\ C_3^+\cos k_1(a+b)+C_3^-\sin k_1(a+b)=C_4^+\cosh k_2(a+b)+C_4^-\sinh k_2(a+b)$$



$$BC2:\ j_{3x}\Big|_{x=a+b}=j_{4x}\Big|_{x=a+b}$$

$$j_{3x}=\frac{\hbar}{2mi}\left(\Psi_3^*\vec{\nabla}\Psi_3-\Psi_3\vec{\nabla}\Psi_3^*\right)=\frac{-k_1}{m}C_3^+\sin k_1x+\frac{k_1}{m}C_3^-\cos k_1x$$

$$j_{4x}=\frac{\hbar}{2mi}\left(\Psi_4^*\vec{\nabla}\Psi_4-\Psi_4\vec{\nabla}\Psi_4^*\right)=\frac{k_2}{m}C_4^+\sinh k_2x+\frac{k_2}{m}C_4^-\cosh k_2x$$

$$\therefore,\ \frac{-k_1}{m}C_3^+\sin k_1(a+b)+\frac{k_1}{m}C_3^-\cos k_1(a+b)=\frac{k_2}{m}C_4^+\sinh k_2(a+b)+\frac{k_2}{m}C_4^-\cosh k_2(a+b)$$

Again, these equations can be represented in matrix form,

$$\begin{bmatrix}\cos k_1(a+b) & \sin k_1(a+b)\\ \frac{-k_1}{m}\sin k_1(a+b) & \frac{k_1}{m}\cos k_1(a+b)\end{bmatrix}\begin{bmatrix}C_3^+\\ C_3^-\end{bmatrix}=\begin{bmatrix}\cosh k_2(a+b) & \sinh k_2(a+b)\\ \frac{k_2}{m}\sinh k_2(a+b) & \frac{k_2}{m}\cosh k_2(a+b)\end{bmatrix}\begin{bmatrix}C_4^+\\ C_4^-\end{bmatrix}$$

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Multiple Quantum Wells – Boundary Conditions (III-IV interface) ($E<V_0$)

(CONTD.)

$$\begin{bmatrix} \cos k_1(a+b) & \sin k_1(a+b) \\ \frac{-k_1}{m}\sin k_1(a+b) & \frac{k_1}{m}\cos k_1(a+b) \end{bmatrix} \begin{bmatrix} C_3^+ \\ C_3^- \end{bmatrix} = \begin{bmatrix} \cosh k_2(a+b) & \sinh k_2(a+b) \\ \frac{k_2}{m}\sinh k_2(a+b) & \frac{k_2}{m}\cosh k_2(a+b) \end{bmatrix} \begin{bmatrix} C_4^+ \\ C_4^- \end{bmatrix}$$

$$solving\ for \begin{bmatrix} C_3^+ \\ C_3^- \end{bmatrix},$$

$$\begin{bmatrix} C_3^+ \\ C_3^- \end{bmatrix} = \frac{m}{k_1} \begin{bmatrix} \frac{k_1}{m}\cos k_1(a+b) & -\sin k_1(a+b) \\ \frac{k_1}{m}\sin k_1(a+b) & \cos k_1(a+b) \end{bmatrix} \begin{bmatrix} \cosh k_2(a+b) & \sinh k_2(a+b) \\ \frac{k_2}{m}\sinh k_2(a+b) & \frac{k_2}{m}\cosh k_2(a+b) \end{bmatrix} \begin{bmatrix} C_4^+ \\ C_4^- \end{bmatrix}$$

x

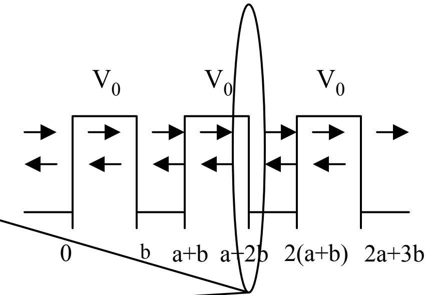
$$\begin{bmatrix} C_3^+ \\ C_3^- \end{bmatrix} = \left[Y_{III-IV} \right] \begin{bmatrix} C_4^+ \\ C_4^- \end{bmatrix}$$

$$Substituting\ this\ in\ Eq.4$$

$$\begin{bmatrix} 1 \\ R \end{bmatrix} = \left[Y_{I-II}\right]\left[Y_{II-III}\right] \begin{bmatrix} C_3^+ \\ C_3^- \end{bmatrix} = \left[Y_{I-II}\right]\left[Y_{II-III}\right]\left[Y_{III-IV}\right] \begin{bmatrix} C_4^+ \\ C_4^- \end{bmatrix} \qquad (Eq.5)$$

This calculation continues iteratively until the carrier crosses through all the interfaces.

Multiple Quantum Wells – Boundary Conditions (IV-V interface) (E<V$_0$)



Using the boundary conditions @ x = a + 2b for regions IV and V we have:

$$BC1: \Psi_4(x=a+2b) = \Psi_3(x=a+2b)$$

$$\therefore, C_4^+ \cosh k_2(a+2b) + C_4^- \sinh k_2(a+2b) = C_5^+ \cos k_1(a+2b) + C_5^- \sin k_1(a+2b)$$

$$BC2: j_{4x}\Big|_{x=a+2b} = j_{5x}\Big|_{x=a+2b}$$

$$j_{4x} = \frac{\hbar}{2mi}\left(\Psi_4^* \vec{\nabla}\Psi_4 - \Psi_4 \vec{\nabla}\Psi_4^*\right) = \frac{k_2}{m} C_4^+ \sinh k_2 x + \frac{k_2}{m} C_4^- \cosh k_2 x$$

$$j_{5x} = \frac{\hbar}{2mi}\left(\Psi_5^* \vec{\nabla}\Psi_5 - \Psi_5 \vec{\nabla}\Psi_5^*\right) = -\frac{k_1}{m} C_5^+ \sin k_5 x + \frac{k_1}{m} C_5^- \cos k_1 x$$

$$\therefore, \frac{k_2}{m} C_4^+ \sinh k_2(a+2b) + \frac{k_2}{m} C_4^- \cosh k_2(a+2b) = -\frac{k_1}{m} C_5^+ \sin k_5(a+2b) + \frac{k_1}{m} C_5^- \cos k_1(a+2b)$$

Again, these equations can be represented in matrix form,

$$\begin{bmatrix} \cosh k_2(a+2b) & \sinh k_2(a+2b) \\ \frac{k_2}{m}\sinh k_2(a+2b) & \frac{k_2}{m}\cosh k_2(a+2b) \end{bmatrix} \begin{bmatrix} C_4^+ \\ C_4^- \end{bmatrix} = \begin{bmatrix} \cos k_1(a+2b) & \sin k_1(a+2b) \\ -\frac{k_1}{m}\sin k_1(a+2b) & \frac{k_1}{m}\cos k_1(a+2b) \end{bmatrix} \begin{bmatrix} C_5^+ \\ C_5^- \end{bmatrix}$$

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Multiple Quantum Wells – Boundary Conditions (IV-V interface) (E<V$_0$)

(CONTD.)

$$\begin{bmatrix} \cosh k_2(a+2b) & \sinh k_2(a+2b) \\ \frac{k_2}{m}\sinh k_2(a+2b) & \frac{k_2}{m}\cosh k_2(a+2b) \end{bmatrix} \begin{bmatrix} C_4^+ \\ C_4^- \end{bmatrix} = \begin{bmatrix} \cos k_1(a+2b) & \sin k_1(a+2b) \\ -\frac{k_1}{m}\sin k_1(a+2b) & \frac{k_1}{m}\cos k_1(a+2b) \end{bmatrix} \begin{bmatrix} C_5^+ \\ C_5^- \end{bmatrix}$$

$$Solving\ \ for \begin{bmatrix} C_4^+ \\ C_4^- \end{bmatrix},$$

$$\begin{bmatrix} C_4^+ \\ C_4^- \end{bmatrix} = \frac{m}{k_2} \begin{bmatrix} \frac{k_2}{m}\cosh k_2(a+2b) & -\sinh k_2(a+2b) \\ -\frac{k_2}{m}\sinh k_2(a+2b) & \cosh k_2(a+2b) \end{bmatrix} \begin{bmatrix} \cos k_1(a+2b) & \sin k_1(a+2b) \\ -\frac{k_1}{m}\sin k_1(a+2b) & \frac{k_1}{m}\cos k_1(a+2b) \end{bmatrix} \begin{bmatrix} C_5^+ \\ C_5^- \end{bmatrix}$$

x

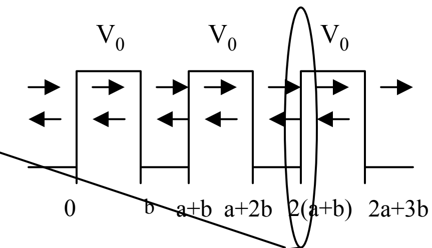
$$\begin{bmatrix} C_4^+ \\ C_4^- \end{bmatrix} = \left[Y_{IV-V}\right] \begin{bmatrix} C_5^+ \\ C_5^- \end{bmatrix}$$

$$Substituting\ the\ above\ \ in\ Eq.5,$$

$$\begin{bmatrix} 1 \\ R \end{bmatrix} = \left[Y_{I-II}\right]\left[Y_{II-III}\right]\left[Y_{III-IV}\right] \begin{bmatrix} C_4^+ \\ C_4^- \end{bmatrix} = \left[Y_{I-II}\right]\left[Y_{II-III}\right]\left[Y_{III-IV}\right]\left[Y_{IV-V}\right] \begin{bmatrix} C_5^+ \\ C_5^- \end{bmatrix} \qquad (Eq.6)$$

Multiple Quantum Wells – Boundary Conditions (V-VI interface) ($E<V_0$)

Using the boundary conditions @ x = 2(a + b) for regions V and VI:



$$BC1: \Psi_5(x=2(a+b)) = \Psi_6(x=2(a+b))$$

$$\therefore, C_5^+ \cos k_1(2a+2b) + C_5^- \sin k_1(2a+2b) = C_6^+ \cosh k_2(2a+2b) + C_6^- \sinh k_2(2a+2b)$$

$$BC2: j_{5x}\Big|_{x=2(a+b)} = j_{6x}\Big|_{x=2(a+b)}$$

$$j_{5x} = \frac{\hbar}{2mi}\left(\Psi_5^* \vec{\nabla}\Psi_5 - \Psi_5 \vec{\nabla}\Psi_5^*\right) = \frac{-k_1}{m} C_5^+ \sin k_1 x + \frac{k_1}{m} C_5^- \cos k_1 x$$

$$j_{6x} = \frac{\hbar}{2mi}\left(\Psi_6^* \vec{\nabla}\Psi_6 - \Psi_6 \vec{\nabla}\Psi_6^*\right) = \frac{k_2}{m} C_6^+ \sinh k_2 x + \frac{k_2}{m} C_6^- \cosh k_2 x$$

$$\therefore, \frac{-k_1}{m} C_5^+ \sin k_1(2a+2b) + \frac{k_1}{m} C_5^- \cos k_1(2a+2b) = \frac{k_2}{m} C_6^+ \sinh k_2(2a+2b) + \frac{k_2}{m} C_6^- \cosh k_2(2a+2b)$$

Again, these equations can be expressed in matrix form,

$$\begin{bmatrix} \cos k_1(2a+2b) & \sin k_1(2a+2b) \\ \frac{-k_1}{m}\sin k_1(2a+2b) & \frac{k_1}{m}\cos k_1(2a+2b) \end{bmatrix} \begin{bmatrix} C_5^+ \\ C_5^- \end{bmatrix} = \begin{bmatrix} \cosh k_2(2a+2b) & \sinh k_2(2a+2b) \\ \frac{k_2}{m}\sinh k_2(2a+2b) & \frac{k_2}{m}\cosh k_2(2a+2b) \end{bmatrix} \begin{bmatrix} C_6^+ \\ C_6^- \end{bmatrix}$$

(CONTD……..)

Multiple Quantum Wells – Boundary Conditions (V-VI interface) ($E<V_0$)

(CONTD.)

$$\begin{bmatrix} \cos k_1(2a+2b) & \sin k_1(2a+2b) \\ \frac{-k_1}{m}\sin k_1(2a+2b) & \frac{k_1}{m}\cos k_1(2a+2b) \end{bmatrix}\begin{bmatrix} C_5^+ \\ C_5^- \end{bmatrix} = \begin{bmatrix} \cosh k_2(2a+2b) & \sinh k_2(2a+2b) \\ \frac{k_2}{m}\sinh k_2(2a+2b) & \frac{k_2}{m}\cosh k_2(2a+2b) \end{bmatrix}\begin{bmatrix} C_6^+ \\ C_6^- \end{bmatrix}$$

$$\text{Solving for} \begin{bmatrix} C_5^+ \\ C_5^- \end{bmatrix},$$

$$\begin{bmatrix} C_5^+ \\ C_5^- \end{bmatrix} = \frac{m}{k_1}\begin{bmatrix} \frac{k_1}{m}\cos k_1(2a+2b) & -\sin k_1(2a+2b) \\ \frac{k_1}{m}\sin k_1(2a+2b) & \cos k_1(2a+2b) \end{bmatrix}\begin{bmatrix} \cosh k_2(2a+2b) & \sinh k_2(2a+2b) \\ \frac{k_2}{m}\sinh k_2(2a+2b) & \frac{k_2}{m}\cosh k_2(2a+2b) \end{bmatrix}\begin{bmatrix} C_6^+ \\ C_6^- \end{bmatrix}$$

x

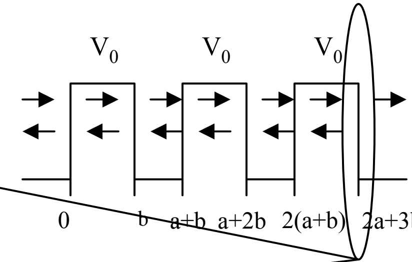
$$\begin{bmatrix} C_5^+ \\ C_5^- \end{bmatrix} = \left[Y_{V-VI}\right]\begin{bmatrix} C_6^+ \\ C_6^- \end{bmatrix}$$

Substituting the above in Eq.6,

$$\begin{bmatrix} 1 \\ R \end{bmatrix} = \left[Y_{I-II}\right]\left[Y_{II-III}\right]\left[Y_{III-IV}\right]\left[Y_{IV-V}\right]\begin{bmatrix} C_5^+ \\ C_5^- \end{bmatrix} = \left[Y_{I-II}\right]\left[Y_{II-III}\right]\left[Y_{III-IV}\right]\left[Y_{IV-V}\right]\left[Y_{V-VI}\right]\begin{bmatrix} C_6^+ \\ C_6^- \end{bmatrix} \qquad (Eq.7)$$

Multiple Quantum Wells – Boundary Conditions (VI-VII interface) (E<V$_0$)

Lastly, using the boundary conditions @ x = 2a + 3b for regions VI and VII:



$$BC\,1:\ \Psi_6(x=2a+3b)=\Psi_7(x=2a+3b)$$

$$\therefore,\ C_6^+\cosh k_2(2a+3b)+C_6^-\sinh k_2(2a+3b)=Te^{ik_1b}$$

$$BC\,2:\ j_{6x}\Big|_{x=(2a+3b)}=j_{7x}\Big|_{x=(2a+3b)}$$

$$j_{6x}=\frac{\hbar}{2mi}\left(\Psi_6^*\,\vec{\nabla}\Psi_6-\Psi_6\,\vec{\nabla}\Psi_6^*\right)=\frac{k_2}{m}C_6^+\sinh k_2x+\frac{k_2}{m}C_6^-\cosh k_2x$$

$$j_{7x}=\frac{\hbar}{2mi}\left(\Psi_7^*\,\vec{\nabla}\Psi_7-\Psi_7\,\vec{\nabla}\Psi_7^*\right)=ik_1Te^{ik_1x}$$

$$\therefore,\ \frac{k_2}{m}C_6^+\sinh k_2(2a+3b)+\frac{k_2}{m}C_6^-\cosh k_2(2a+3b)=ik_1Te^{ik_1b}$$

Again, these equations can be represented in matrix form,

$$\begin{bmatrix}\cosh k_2(2a+3b) & \sinh k_2(2a+3b)\\ \frac{k_2}{m}\sinh k_2(2a+3b) & \frac{k_2}{m}\cosh k_2(2a+3b)\end{bmatrix}\begin{bmatrix}C_6^+\\ C_6^-\end{bmatrix}=\begin{bmatrix}Te^{ik_1b} & 0\\ ik_1Te^{ik_1b} & 0\end{bmatrix}\begin{bmatrix}T\\ 0\end{bmatrix}$$

(CONTD……)

Multiple Quantum Wells – Boundary Conditions (VI-VII interface) ($E<V_0$)

(CONTD.)

$$\begin{bmatrix} \cosh k_2(2a+3b) & \sinh k_2(2a+3b) \\ \frac{k_2}{m}\sinh k_2(2a+3b) & \frac{k_2}{m}\cosh k_2(2a+3b) \end{bmatrix}\begin{bmatrix} C_6^+ \\ C_6^- \end{bmatrix} = \begin{bmatrix} Te^{ik_1b} & 0 \\ ik_1Te^{ik_1b} & 0 \end{bmatrix}\begin{bmatrix} T \\ 0 \end{bmatrix}$$

$$\textit{Solving for}\begin{bmatrix} C_6^+ \\ C_6^- \end{bmatrix},$$

$$\begin{bmatrix} C_6^+ \\ C_6^- \end{bmatrix} = \frac{m}{k_2}\begin{bmatrix} \frac{k_2}{m}\cosh k_2(2a+3b) & -\sinh k_2(2a+3b) \\ -\frac{k_2}{m}\sinh k_2(2a+3b) & \cosh k_2(2a+3b) \end{bmatrix}\begin{bmatrix} e^{ik_1b} & 0 \\ ik_1e^{ik_1b} & 0 \end{bmatrix}\begin{bmatrix} T \\ 0 \end{bmatrix}$$

x

$$\begin{bmatrix} C_6^+ \\ C_6^- \end{bmatrix} = \left[Y_{VI-VII}\right]\begin{bmatrix} T \\ 0 \end{bmatrix}$$

$$\textit{Subsituting the above in Eq.7,}$$

$$\begin{bmatrix} 1 \\ R \end{bmatrix} = \left[Y_{I-II}\right]\left[Y_{II-III}\right]\left[Y_{III-IV}\right]\left[Y_{IV-V}\right]\left[Y_{V-VI}\right]\begin{bmatrix} C_6^+ \\ C_6^- \end{bmatrix} = \left[Y_{I-II}\right]\left[Y_{II-III}\right]\left[Y_{III-IV}\right]\left[Y_{IV-V}\right]\left[Y_{V-VI}\right]\left[Y_{VI-VII}\right]\begin{bmatrix} T \\ 0 \end{bmatrix} \qquad (Eq.\ 8)$$

We have finally got a useful relationship between the reflection co-efficient and transmission co-efficient. However, multiplying the transfer matrices is very complex as will be seen from the next slide.

(CONTD…….)

Multiple Quantum Wells – Final Matrix (E<V$_0$)

The complexity in multiplying these transfer matrices can be seen from the final matrix below:

$$\begin{bmatrix} 1 \\ R \end{bmatrix} = [Y_{I-II}][Y_{II-III}][Y_{III-IV}][Y_{IV-V}][Y_{V-VI}][Y_{VI-VII}]\begin{bmatrix} \tau \\ 0 \end{bmatrix} \quad (Eq.8)$$

$$\begin{bmatrix} 1 \\ R \end{bmatrix} = \left.\begin{bmatrix} \frac{1}{2} & \frac{m}{2ik_1} \\ \frac{1}{2} & \frac{-m}{2ik_1} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{k_2}{m} \end{bmatrix}\right\} \mathbf{Y_{I\text{-}II}}$$

$$\left.\times \frac{m}{k_2}\begin{bmatrix} \cosh k_2 b & -\frac{m}{k_2}\sinh k_2 b \\ -\sinh k_2 b & \frac{m}{k_2}\cosh k_2 b \end{bmatrix}\begin{bmatrix} \cos k_1 b & \sin k_1 b \\ -\frac{k_1}{m}\sin k_1 b & \frac{k_1}{m}\cos k_1 b \end{bmatrix}\right\} \mathbf{Y_{II\text{-}III}}$$

$$\left.\times \frac{m}{k_1}\begin{bmatrix} \cos k_1(a+b) & -\frac{m}{k_1}\sin k_1(a+b) \\ \sin k_1(a+b) & \frac{m}{k_1}\cos k_1(a+b) \end{bmatrix}\begin{bmatrix} \cosh k_2(a+b) & \sinh k_2(a+b) \\ \frac{k_2}{m}\sinh k_2(a+b) & \frac{k_2}{m}\cosh k_2(a+b) \end{bmatrix}\right\} \mathbf{Y_{III\text{-}IV}}$$

$$\left.\times \frac{m}{k_2}\begin{bmatrix} \frac{k_2}{m}\cosh k_2(a+2b) & -\sinh k_2(a+2b) \\ -\frac{k_2}{m}\sinh k_2(a+2b) & \cosh k_2(a+2b) \end{bmatrix}\begin{bmatrix} \cos k_1(a+2b) & \sin k_1(a+2b) \\ -\frac{k_1}{m}\sin k_1(a+2b) & \frac{k_1}{m}\cos k_1(a+2b) \end{bmatrix}\right\} \mathbf{Y_{IV\text{-}V}}$$

$$\left.\times \frac{m}{k_1}\begin{bmatrix} \frac{k_1}{m}\cos k_1(2a+2b) & -\sin k_1(2a+2b) \\ \frac{k_1}{m}\sin k_1(2a+2b) & \cos k_1(2a+2b) \end{bmatrix}\begin{bmatrix} \cosh k_2(2a+2b) & \sinh k_2(2a+2b) \\ \frac{k_2}{m}\sinh k_2(2a+2b) & \frac{k_2}{m}\cosh k_2(2a+2b) \end{bmatrix}\right\} \mathbf{Y_{V\text{-}VI}}$$

$$\left.\times \frac{m}{k_2}\begin{bmatrix} \frac{k_2}{m}\cosh k_2(2a+3b) & -\sinh k_2(2a+3b) \\ -\frac{k_2}{m}\sinh k_2(2a+3b) & \cosh k_2(2a+3b) \end{bmatrix}\begin{bmatrix} e^{ik_1 b} & 0 \\ ik_1 e^{ik_1 b} & 0 \end{bmatrix}\begin{bmatrix} T \\ 0 \end{bmatrix}\right\} \mathbf{Y_{VI\text{-}VII}}$$

Multiple Quantum Wells – Reflection and Transmission Probabilities ($E<V_0$)

(CONTD.)

$$\begin{bmatrix} 1 \\ R \end{bmatrix} = \left[Y_{I-II}\right]\left[Y_{II-III}\right]\left[Y_{III-IV}\right]\left[Y_{IV-V}\right]\left[Y_{V-VI}\right]\left[Y_{VI-VII}\right]\begin{bmatrix} \tau \\ 0 \end{bmatrix} \qquad (Eq.8)$$

Equation 8 can now be used to calculate the transmission and reflection co-efficients. We would need a powerful math tool in order to multiply the complicated transfer matrices. After this multiplication is performed, we would obtain a 2x2 matrix as shown below:

$$\boxed{\begin{bmatrix} 1 \\ R \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} \tau \\ 0 \end{bmatrix}} \qquad (Eq.9)$$

The ***Transmission Co-efficient, "T",*** can then be found by using:

$$\boxed{1 = AT \Rightarrow T = \frac{1}{A}}$$

The ***Reflection Co-efficient, "R",*** would then be given by:

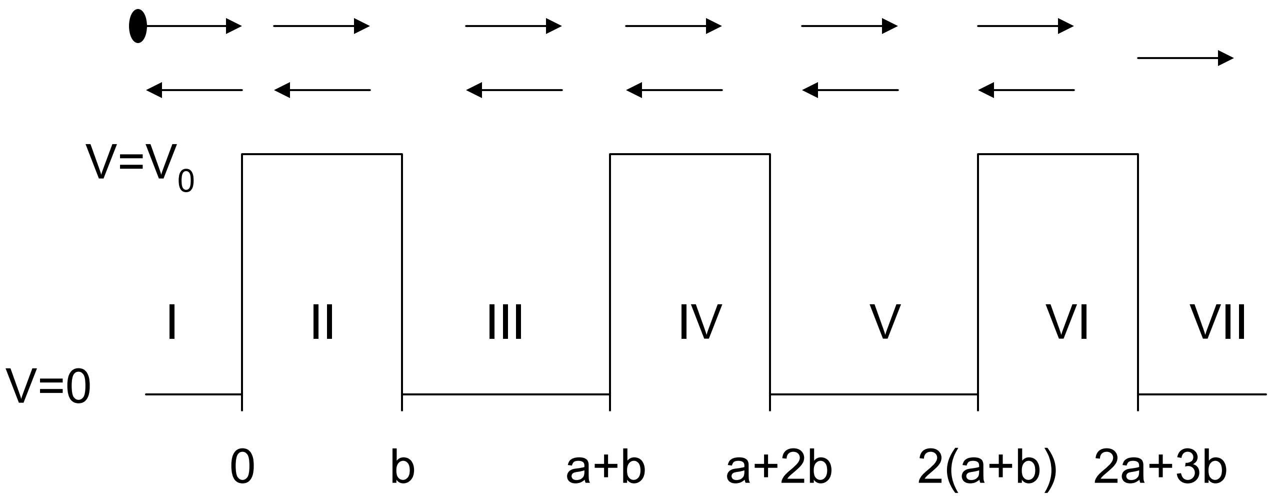
$$\boxed{R = CT = \frac{C}{A}}$$

Therefore, the ***Transmission Probability and Reflection Probability*** is given by:

$$\boxed{TT^* = \left(\frac{1}{A}\right)\left(\frac{1}{A}\right)^* \qquad RR^* = 1 - TT^* = 1 - \left(\frac{1}{A}\right)\left(\frac{1}{A}\right)^* \quad OR \quad RR^* = \left(\frac{C}{A}\right)\left(\frac{C}{A}\right)^*}$$

Multiple Quantum Wells – General Structure ($E>V_0$)

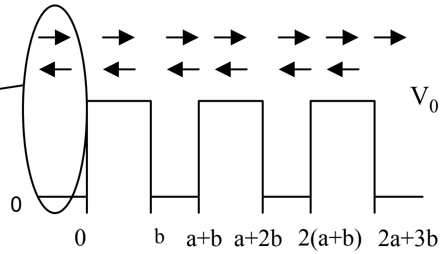
Now, let us consider the case when an electron with energy E greater than V_0 is incident on the structure:



Again, we take the same approach as we did earlier: we solve for the wave function in each region and then we apply the boundary conditions between the regions and finally, solve for the reflection and transmission probabilities.

Multiple Quantum Wells – Wave Functions Review (E>V_0)

For region I, the wavefunction is the same as the E<V_0 case as there is no potential involved here:



$$\frac{-\hbar^2}{2m}\frac{d^2\Psi_1}{dx^2} + \cancelto{0}{V_0}\Psi_1 = E\Psi_1$$

$$\Rightarrow \Psi_1 = \underbrace{e^{ik_1x}}_{\textbf{Incident wave}} + \underbrace{R\,e^{-ik_1x}}_{\textbf{Reflected wave}}$$

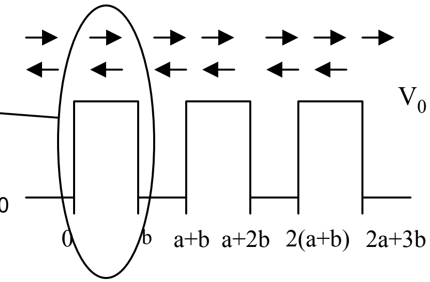
where,

$$k_1 = \sqrt{\frac{2mE}{\hbar^2}}$$

$$R = reflection\ co-efficient$$

Multiple Quantum Wells – Wave Functions Review (E>V_0)

For region II, we have a potential of V_0:



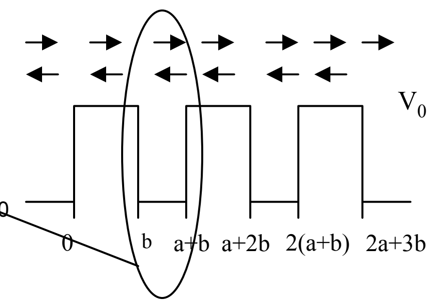
$$\frac{-\hbar^2}{2m}\frac{d^2\Psi_2}{dx^2} + V_0\Psi_2 = E\Psi_2$$

$$\Rightarrow \Psi_2 = C_2^+ e^{ik_2 x} + C_2^- e^{-ik_2 x} \quad \text{where,} \quad k_2 = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$$

+x direction wave **-x direction wave**

Or, in terms of cosine and sine functions, $\Psi_2 = C_2^+ \cos k_2 x + C_2^- \sin k_2 x$

For region III,



$$\frac{-\hbar^2}{2m}\frac{d^2\Psi_3}{dx^2} + \cancelto{0}{V_0}\Psi_3 = E\Psi_3$$

$$\Psi_3 = C_3^+ e^{ik_1 x} + C_3^- e^{-ik_1 x} \quad \text{where,} \quad k_1 = \sqrt{\frac{2mE}{\hbar^2}}$$

Or,

$$\Psi_3 = C_3^+ \cos k_1 x + C_3^- \sin k_1 x$$

+x direction wave **-x direction wave**

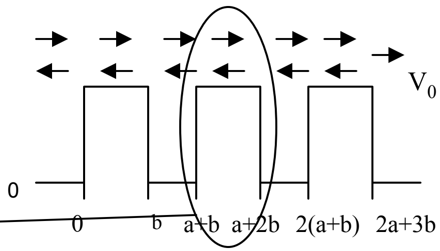
Similarly, we can solve for the wave functions for the other regions with each region with a potential having similar solutions as Ψ_2 and the regions without a potential having solutions similar to Ψ_3

Multiple Quantum Wells – Wave Functions Review (E>V_0)

For Region IV:

$$\Psi_4 = C_4^+ e^{ik_2 x} + C_4^- e^{-ik_2 x}, \qquad k_2 = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$$

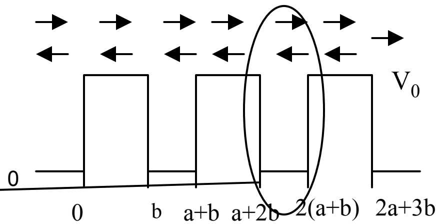
$$\Psi_4 = C_4^+ \cos k_2 x + C_4^- \sin k_2 x$$



For Region V:

$$\Psi_5 = C_5^+ e^{ik_1 x} + C_5^- e^{-ik_1 x}, \qquad k_1 = \sqrt{\frac{2mE}{\hbar^2}}$$

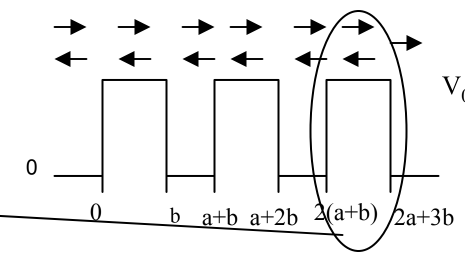
$$\Psi_5 = C_5^+ \cos k_1 x + C_5^- \sin k_1 x$$



For Region VI:

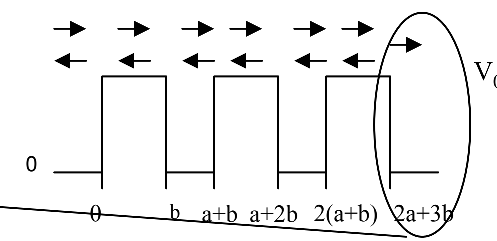
$$\Psi_6 = C_6^+ e^{ik_2 x} + C_6^- e^{-ik_2 x}, \qquad k_2 = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$$

$$\Psi_6 = C_6^+ \cos k_2 x + C_6^- \sin k_2 x$$



For Region VII:

$$\Psi_7 = T e^{ik_1 x}, \qquad k_1 = \sqrt{\frac{2mE}{\hbar^2}}$$

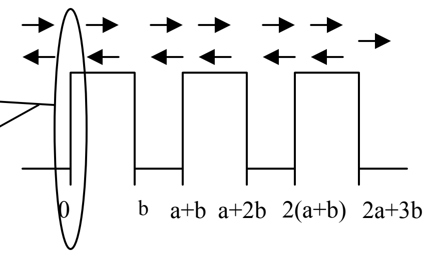


Note: For region VII, only a +x component exists as the wave extends out to infinity and there is no barrier to reflect the wave back

Multiple Quantum Wells – Boundary Conditions (I-II interface) (E>V_0)

Now, we can apply the boundary conditions between each region. For regions I and II, we can use two boundary conditions:

- *Boundary Condition 1 (Continuity of the wave function):*



$$\Psi_1(x=0) = \Psi_2(x=0)$$

$$\left(e^{ik_1x} + R\,e^{-ik_1x}\right)\Big|_{x=0} = \left(C_2^+ \cos k_2x + C_2^- \sin k_2x\right)\Big|_{x=0}$$

$$1 + R = C_2^+ \qquad (Eq.\,10)$$

- *Boundary Condition 2 (Probability Current Density):*

$$j_{1x}\big|_{x=0} = j_{2x}\big|_{x=0}$$

$$j_{1x} = \frac{\hbar}{2mi}\left(\Psi_1^* \vec{\nabla}\Psi_1 - \Psi_1 \vec{\nabla}\Psi_1^*\right) = \left(\frac{ik_1}{m} e^{ik_1x} - \frac{ik_1}{m} R\,e^{-ik_1x}\right)$$

$$j_{2x} = \frac{\hbar}{2mi}\left(\Psi_2^* \vec{\nabla}\Psi_2 - \Psi_2 \vec{\nabla}\Psi_2^*\right) = \left(-\frac{k_2}{m} C_2^+ \sin k_2x + \frac{k_2}{m} C_2^- \cos k_2x\right)$$

$$\therefore, \left(\frac{ik_1}{m} e^{ik_1x} - \frac{ik_1}{m} R\,e^{-ik_1x}\right)\Bigg|_{x=0} = \left(-\frac{k_2}{m} C_2^+ \sin k_2x + \frac{k_2}{m} C_2^- \cos k_2x\right)\Bigg|_{x=0}$$

$$\frac{ik_1}{m} - \frac{ik_1}{m} R = \frac{k_2}{m} C_2^- \qquad (Eq.11)$$

Multiple Quantum Wells – Boundary Conditions (I-II interface) (E>V_0)

But, equation 10 and 11 can also be expressed in matrix form

$$1 + R = C_2^+ \qquad (Eq.\,10)$$

$$\frac{ik_1}{m} - \frac{ik_1}{m}R = \frac{k_2}{m}C_2^- \qquad (Eq.11)$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 1 \\ \frac{ik_1}{m} & \frac{-ik_1}{m} \end{bmatrix} \begin{bmatrix} 1 \\ R \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{k_2}{m} \end{bmatrix} \begin{bmatrix} C_2^+ \\ C_2^- \end{bmatrix}$$

$$Solving\ \ for \begin{bmatrix} 1 \\ R \end{bmatrix},$$

$$\begin{bmatrix} 1 \\ R \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \frac{ik_1}{m} & \frac{-ik_1}{m} \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & \frac{k_2}{m} \end{bmatrix} \begin{bmatrix} C_2^+ \\ C_2^- \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ R \end{bmatrix} = \left(\frac{-m}{2ik_1} \right) \begin{bmatrix} \frac{-ik_1}{m} & -1 \\ \frac{-ik_1}{m} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{k_2}{m} \end{bmatrix} \begin{bmatrix} C_2^+ \\ C_2^- \end{bmatrix}$$

MATRIX IDENTITIES USED HERE:

$1.\,if\ \ AB = C, then\, B = A^{-1}C$

$2.\,if\ \ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},\ then\ \ A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

(CONTD……)

Multiple Quantum Wells – Boundary Conditions (I-II interface) (E>V_0)

(CONTD.)

$$\begin{bmatrix} 1 \\ R \end{bmatrix} = \left(\frac{-m}{2ik_1}\right) \begin{bmatrix} \dfrac{-ik_1}{m} & -1 \\ \dfrac{-ik_1}{m} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{k_2}{m} \end{bmatrix} \begin{bmatrix} C_2^+ \\ C_2^- \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ R \end{bmatrix} = \underbrace{\begin{bmatrix} \dfrac{1}{2} & \dfrac{m}{2ik_1} \\ \dfrac{1}{2} & \dfrac{-m}{2ik_1} \end{bmatrix}}_{\textbf{[a]}} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & \dfrac{k_2}{m} \end{bmatrix}}_{\textbf{[b]}} \begin{bmatrix} C_2^+ \\ C_2^- \end{bmatrix}$$

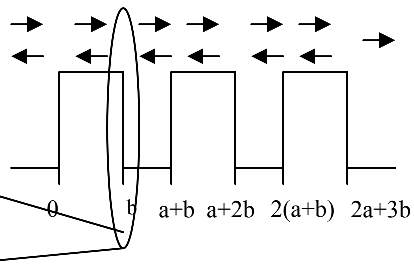
[a] **x** **[b]**

$$\begin{bmatrix} 1 \\ R \end{bmatrix} = \left[Y_{I-II}\right] \begin{bmatrix} C_2^+ \\ C_2^- \end{bmatrix} \qquad (Eq.\,12)$$

where, Y_{I-II} is the ***transfer matrix*** for the region I-II interface

Multiple Quantum Wells – Boundary Conditions (II-III interface) ($E>V_0$)

We can follow the same procedure we did for the regions I-II interface for obtaining the transfer matrix for the region II-III. Again, we start by using the boundary conditions @ x = b for regions II and III:



$$BC1: \Psi_2(x=b) = \Psi_3(x=b)$$

$$\therefore, C_2^+ \cos k_2 b + C_2^- \sin k_2 b = C_3^+ \cos k_1 b + C_3^- \sin k_1 b$$

$$BC2: j_{2x}\big|_{x=b} = j_{3x}\big|_{x=b}$$

$$j_{2x} = \frac{\hbar}{2mi}\left(\Psi_2^* \vec{\nabla}\Psi_2 - \Psi_2 \vec{\nabla}\Psi_2^*\right) = -\frac{k_2}{m} C_2^+ \sin k_2 x + \frac{k_2}{m} C_2^- \cos k_2 x$$

$$j_{3x} = \frac{\hbar}{2mi}\left(\Psi_3^* \vec{\nabla}\Psi_3 - \Psi_3 \vec{\nabla}\Psi_3^*\right) = -\frac{k_1}{m} C_3^+ \sin k_1 x + \frac{k_1}{m} C_3^- \cos k_1 x$$

$$\therefore, -\frac{k_2}{m} C_2^+ \sin k_2 b + \frac{k_2}{m} C_2^- \cos k_2 b = -\frac{k_1}{m} C_3^+ \sin k_1 b + \frac{k_1}{m} C_3^- \cos k_1 b$$

Again, we can represent these two equations in in the matrix form:

$$\begin{bmatrix} \cos k_2 b & \sin k_2 b \\ -\frac{k_2}{m}\sin k_2 b & \frac{k_2}{m}\cos k_2 b \end{bmatrix} \begin{bmatrix} C_2^+ \\ C_2^- \end{bmatrix} = \begin{bmatrix} \cos k_1 b & \sin k_1 b \\ -\frac{k_1}{m}\sin k_1 b & \frac{k_1}{m}\cos k_1 b \end{bmatrix} \begin{bmatrix} C_3^+ \\ C_3^- \end{bmatrix}$$

(CONTD……)

Multiple Quantum Wells – Boundary Conditions (II-III interface) (E>V_0)

(CONTD.)

$$\begin{bmatrix} \cos k_2 b & \sin k_2 b \\ -\frac{k_2}{m}\sin k_2 b & \frac{k_2}{m}\cos k_2 b \end{bmatrix} \begin{bmatrix} C_2^+ \\ C_2^- \end{bmatrix} = \begin{bmatrix} \cos k_1 b & \sin k_1 b \\ -\frac{k_1}{m}\sin k_1 b & \frac{k_1}{m}\cos k_1 b \end{bmatrix} \begin{bmatrix} C_3^+ \\ C_3^- \end{bmatrix}$$

$$\text{Solving for} \begin{bmatrix} C_2^+ \\ C_2^- \end{bmatrix},$$

$$\begin{bmatrix} C_2^+ \\ C_2^- \end{bmatrix} = \frac{m}{k_2} \begin{bmatrix} \frac{k_2}{m}\cos k_2 b & -\sin k_2 b \\ \frac{k_2}{m}\sin k_2 b & \cos k_2 b \end{bmatrix} \begin{bmatrix} \cos k_1 b & \sin k_1 b \\ -\frac{k_1}{m}\sin k_1 b & \frac{k_1}{m}\cos k_1 b \end{bmatrix} \begin{bmatrix} C_3^+ \\ C_3^- \end{bmatrix}$$

x

$$\begin{bmatrix} C_2^+ \\ C_2^- \end{bmatrix} = \left[Y_{II-III}\right] \begin{bmatrix} C_3^+ \\ C_3^- \end{bmatrix}$$

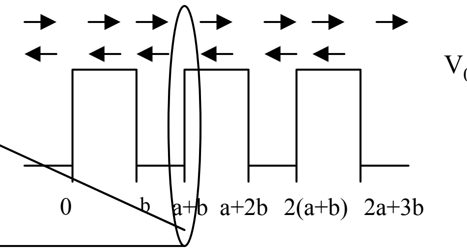
Substituting the above in Eq.12, we have

$$\begin{bmatrix} 1 \\ R \end{bmatrix} = \left[Y_{I-II}\right] \begin{bmatrix} C_2^+ \\ C_2^- \end{bmatrix} = \left[Y_{I-II}\right]\left[Y_{II-III}\right] \begin{bmatrix} C_3^+ \\ C_3^- \end{bmatrix} \qquad (Eq.\ 13)$$

Note: Now, it should be obvious to see why we use the $Y_{\#1-\#2}$ notation - we avoid rewriting the rather nasty matrix multiplication with this notation.

Multiple Quantum Wells – Boundary Conditions (III-IV interface) ($E>V_0$)

We continue finding the transfer matrices. Using the boundary conditions @ x = a + b for regions III and IV:



$$BC1: \Psi_3(x=a+b)=\Psi_4(x=a+b)$$

$$\therefore, C_3^+\cos k_1(a+b)+C_3^-\sin k_1(a+b)=C_4^+\cos k_2(a+b)+C_4^-\sin k_2(a+b)$$

$$BC2: j_{3x}\big|_{x=a+b}=j_{4x}\big|_{x=a+b}$$

$$j_{3x}=\frac{\hbar}{2mi}\left(\Psi_3^*\vec{\nabla}\Psi_3-\Psi_3\vec{\nabla}\Psi_3^*\right)=-\frac{k_1}{m}C_3^+\sin k_1x+\frac{k_1}{m}C_3^-\cos k_1x$$

$$j_{4x}=\frac{\hbar}{2mi}\left(\Psi_4^*\vec{\nabla}\Psi_4-\Psi_4\vec{\nabla}\Psi_4^*\right)=-\frac{k_2}{m}C_4^+\sin k_2x+\frac{k_2}{m}C_4^-\cos k_2x$$

$$\therefore,-\frac{k_1}{m}C_3^+\sin k_1(a+b)+\frac{k_1}{m}C_3^-\cos k_1(a+b)=-\frac{k_2}{m}C_4^+\sin k_2(a+b)+\frac{k_2}{m}C_4^-\cos k_2(a+b)$$

Again, these equations can be represented in matrix form,

$$\begin{bmatrix}\cos k_1(a+b) & \sin k_1(a+b)\\ -\frac{k_1}{m}\sin k_1(a+b) & \frac{k_1}{m}\cos k_1(a+b)\end{bmatrix}\begin{bmatrix}C_3^+\\ C_3^-\end{bmatrix}=\begin{bmatrix}\cos k_2(a+b) & \sin k_2(a+b)\\ -\frac{k_2}{m}\sin k_2(a+b) & \frac{k_2}{m}\cos k_2(a+b)\end{bmatrix}\begin{bmatrix}C_4^+\\ C_4^-\end{bmatrix}$$

(CONTD………)

Multiple Quantum Wells – Boundary Conditions (III-IV interface) (E>V_0)

(CONTD.)

$$\begin{bmatrix} \cos k_1(a+b) & \sin k_1(a+b) \\ -\frac{k_1}{m}\sin k_1(a+b) & \frac{k_1}{m}\cos k_1(a+b) \end{bmatrix} \begin{bmatrix} C_3^+ \\ C_3^- \end{bmatrix} = \begin{bmatrix} \cos k_2(a+b) & \sin k_2(a+b) \\ -\frac{k_2}{m}\sin k_2(a+b) & \frac{k_2}{m}\cos k_2(a+b) \end{bmatrix} \begin{bmatrix} C_4^+ \\ C_4^- \end{bmatrix}$$

$$\textit{Solving for} \begin{bmatrix} C_4^+ \\ C_4^- \end{bmatrix},$$

$$\begin{bmatrix} C_3^+ \\ C_3^- \end{bmatrix} = \frac{m}{k_1} \begin{bmatrix} \frac{k_1}{m}\cos k_1(a+b) & -\sin k_1(a+b) \\ \frac{k_1}{m}\sin k_1(a+b) & \cos k_1(a+b) \end{bmatrix} \begin{bmatrix} \cos k_2(a+b) & \sin k_2(a+b) \\ -\frac{k_2}{m}\sin k_2(a+b) & \frac{k_2}{m}\cos k_2(a+b) \end{bmatrix} \begin{bmatrix} C_4^+ \\ C_4^- \end{bmatrix}$$

x

$$\begin{bmatrix} C_3^+ \\ C_3^- \end{bmatrix} = \left[Y_{III-IV}\right] \begin{bmatrix} C_4^+ \\ C_4^- \end{bmatrix}$$

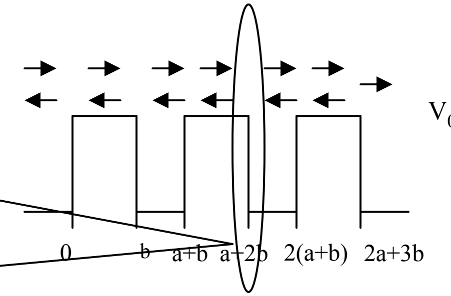
Substituting the above in Eq.13,

$$\begin{bmatrix} 1 \\ R \end{bmatrix} = \left[Y_{I-II}\right]\left[Y_{II-III}\right] \begin{bmatrix} C_3^+ \\ C_3^- \end{bmatrix} = \left[Y_{I-II}\right]\left[Y_{II-III}\right]\left[Y_{III-IV}\right] \begin{bmatrix} C_4^+ \\ C_4^- \end{bmatrix} \qquad (Eq.14)$$

This calculation continues iteratively until the carrier crosses through all the interfaces.

Multiple Quantum Wells – Boundary Conditions (IV-V interface) (E>V_0)

Using the boundary conditions @ x = a + 2b for regions IV and V we have:



$$BC1: \Psi_4(x=a+2b) = \Psi_3(x=a+2b)$$

$$\therefore, C_4^+ \cos k_2(a+2b) + C_4^- \sin k_2(a+2b) = C_5^+ \cos k_1(a+2b) + C_5^- \sin k_1(a+2b)$$

$$BC2: j_{4x}\Big|_{x=a+2b} = j_{5x}\Big|_{x=a+2b}$$

$$j_{4x} = \frac{\hbar}{2mi}\left(\Psi_4^* \vec{\nabla}\Psi_4 - \Psi_4 \vec{\nabla}\Psi_4^*\right) = -\frac{k_2}{m} C_4^+ \sin k_2 x + \frac{k_2}{m} C_4^- \cos k_2 x$$

$$j_{5x} = \frac{\hbar}{2mi}\left(\Psi_5^* \vec{\nabla}\Psi_5 - \Psi_5 \vec{\nabla}\Psi_5^*\right) = -\frac{k_1}{m} C_5^+ \sin k_5 x + \frac{k_1}{m} C_5^- \cos k_1 x$$

$$\therefore, -\frac{k_2}{m} C_4^+ \sin k_2(a+2b) + \frac{k_2}{m} C_4^- \cos k_2(a+2b) = -\frac{k_1}{m} C_5^+ \sin k_5(a+2b) + \frac{k_1}{m} C_5^- \cos k_1(a+2b)$$

Again, these equations can be represented in matrix form,

$$\begin{bmatrix} \cos k_2(a+2b) & \sin k_2(a+2b) \\ -\frac{k_2}{m}\sinh k_2(a+2b) & \frac{k_2}{m}\cos k_2(a+2b) \end{bmatrix} \begin{bmatrix} C_4^+ \\ C_4^- \end{bmatrix} = \begin{bmatrix} \cos k_1(a+2b) & \sin k_1(a+2b) \\ -\frac{k_1}{m}\sin k_1(a+2b) & \frac{k_1}{m}\cos k_1(a+2b) \end{bmatrix} \begin{bmatrix} C_5^+ \\ C_5^- \end{bmatrix}$$

(CONTD……..)

Multiple Quantum Wells – Boundary Conditions (IV-V interface) (E>V$_0$)

(CONTD.)

$$\begin{bmatrix} \cos k_2(a+2b) & \sin k_2(a+2b) \\ -\frac{k_2}{m}\sinh k_2(a+2b) & \frac{k_2}{m}\cos k_2(a+2b) \end{bmatrix} \begin{bmatrix} C_4^+ \\ C_4^- \end{bmatrix} = \begin{bmatrix} \cos k_1(a+2b) & \sin k_1(a+2b) \\ -\frac{k_1}{m}\sin k_1(a+2b) & \frac{k_1}{m}\cos k_1(a+2b) \end{bmatrix} \begin{bmatrix} C_5^+ \\ C_5^- \end{bmatrix}$$

$$Solving\ \ for \begin{bmatrix} C_4^+ \\ C_4^- \end{bmatrix},$$

$$\begin{bmatrix} C_4^+ \\ C_4^- \end{bmatrix} = \frac{m}{k_2} \begin{bmatrix} \frac{k_2}{m}\cos k_2(a+2b) & -\sin k_2(a+2b) \\ \frac{k_2}{m}\sinh k_2(a+2b) & \cos k_2(a+2b) \end{bmatrix} \begin{bmatrix} \cos k_1(a+2b) & \sin k_1(a+2b) \\ -\frac{k_1}{m}\sin k_1(a+2b) & \frac{k_1}{m}\cos k_1(a+2b) \end{bmatrix} \begin{bmatrix} C_5^+ \\ C_5^- \end{bmatrix}$$

x

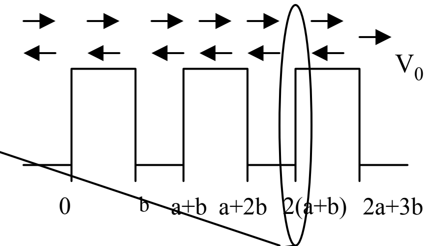
$$\begin{bmatrix} C_4^+ \\ C_4^- \end{bmatrix} = \left[Y_{IV-V}\right] \begin{bmatrix} C_5^+ \\ C_5^- \end{bmatrix}$$

$$Substituting\ the\ above\ in\ Eq.14,$$

$$\begin{bmatrix} 1 \\ R \end{bmatrix} = \left[Y_{I-II}\right]\left[Y_{II-III}\right]\left[Y_{III-IV}\right] \begin{bmatrix} C_4^+ \\ C_4^- \end{bmatrix} = \left[Y_{I-II}\right]\left[Y_{II-III}\right]\left[Y_{III-IV}\right]\left[Y_{IV-V}\right] \begin{bmatrix} C_5^+ \\ C_5^- \end{bmatrix} \qquad (Eq.15)$$

Multiple Quantum Wells – Boundary Conditions (V-VI interface) (E>V_0)

Using the boundary conditions @ x = 2(a + b) for regions V and VI:



$$BC1: \Psi_5(x=2(a+b)) = \Psi_6(x=2(a+b))$$

$$\therefore, C_5^+ \cos k_1(2a+2b) + C_5^- \sin k_1(2a+2b) = C_6^+ \cos k_2(2a+2b) + C_6^- \sin k_2(2a+2b)$$

$$BC2: j_{5x}\Big|_{x=2(a+b)} = j_{6x}\Big|_{x=2(a+b)}$$

$$j_{5x} = \frac{\hbar}{2mi}\left(\Psi_5^* \vec{\nabla}\Psi_5 - \Psi_5 \vec{\nabla}\Psi_5^*\right) = -\frac{k_1}{m} C_5^+ \sin k_1 x + \frac{k_1}{m} C_5^- \cos k_1 x$$

$$j_{6x} = \frac{\hbar}{2mi}\left(\Psi_6^* \vec{\nabla}\Psi_6 - \Psi_6 \vec{\nabla}\Psi_6^*\right) = -\frac{k_2}{m} C_6^+ \sin k_2 x + \frac{k_2}{m} C_6^- \cosh k_2 x$$

$$\therefore, -\frac{k_1}{m} C_5^+ \sin k_1(2a+2b) + \frac{k_1}{m} C_5^- \cos k_1(2a+2b) = -\frac{k_2}{m} C_6^+ \sin k_2(2a+2b) + \frac{k_2}{m} C_6^- \cos k_2(2a+2b)$$

Again, these equations can be expressed in matrix form,

$$\begin{bmatrix} \cos k_1(2a+2b) & \sin k_1(2a+2b) \\ -\frac{k_1}{m}\sin k_1(2a+2b) & \frac{k_1}{m}\cos k_1(2a+2b) \end{bmatrix} \begin{bmatrix} C_5^+ \\ C_5^- \end{bmatrix} = \begin{bmatrix} \cos k_2(2a+2b) & \sin k_2(2a+2b) \\ -\frac{k_2}{m}\sin k_2(2a+2b) & \frac{k_2}{m}\cos k_2(2a+2b) \end{bmatrix} \begin{bmatrix} C_6^+ \\ C_6^- \end{bmatrix}$$

(CONTD……..)

Multiple Quantum Wells – Boundary Conditions (V-VI interface) (E>V_0)

(CONTD.)

$$\begin{bmatrix} \cos k_1(2a+2b) & \sin k_1(2a+2b) \\ -\frac{k_1}{m}\sin k_1(2a+2b) & \frac{k_1}{m}\cos k_1(2a+2b) \end{bmatrix} \begin{bmatrix} C_5^+ \\ C_5^- \end{bmatrix} = \begin{bmatrix} \cos k_2(2a+2b) & \sin k_2(2a+2b) \\ -\frac{k_2}{m}\sin k_2(2a+2b) & \frac{k_2}{m}\cos k_2(2a+2b) \end{bmatrix} \begin{bmatrix} C_6^+ \\ C_6^- \end{bmatrix}$$

$$\textit{Solving for} \begin{bmatrix} C_5^+ \\ C_5^- \end{bmatrix},$$

$$\begin{bmatrix} C_5^+ \\ C_5^- \end{bmatrix} = \frac{m}{k_1} \begin{bmatrix} \frac{k_1}{m}\cos k_1(2a+2b) & -\sin k_1(2a+2b) \\ \frac{k_1}{m}\sin k_1(2a+2b) & \cos k_1(2a+2b) \end{bmatrix} \begin{bmatrix} \cos k_2(2a+2b) & \sin k_2(2a+2b) \\ -\frac{k_2}{m}\sin k_2(2a+2b) & \frac{k_2}{m}\cos k_2(2a+2b) \end{bmatrix} \begin{bmatrix} C_6^+ \\ C_6^- \end{bmatrix}$$

x

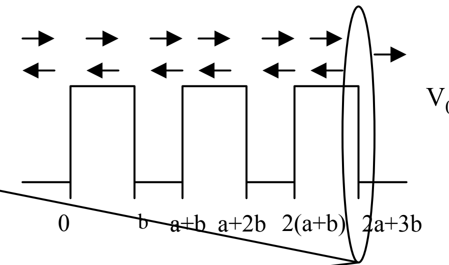
$$\begin{bmatrix} C_5^+ \\ C_5^- \end{bmatrix} = \left[Y_{V-VI} \right] \begin{bmatrix} C_6^+ \\ C_6^- \end{bmatrix}$$

*Substituting the above in Eq.*15,

$$\begin{bmatrix} 1 \\ R \end{bmatrix} = \left[Y_{I-II}\right]\left[Y_{II-III}\right]\left[Y_{III-IV}\right]\left[Y_{IV-V}\right] \begin{bmatrix} C_5^+ \\ C_5^- \end{bmatrix} = \left[Y_{I-II}\right]\left[Y_{II-III}\right]\left[Y_{III-IV}\right]\left[Y_{IV-V}\right]\left[Y_{V-VI}\right] \begin{bmatrix} C_6^+ \\ C_6^- \end{bmatrix} \qquad (Eq.16)$$

Multiple Quantum Wells – Boundary Conditions (VI-VII interface) (E>V_0)

Lastly, using the boundary conditions @ x = 2a + 3b for regions VI and VII:



$$BC\,1: \Psi_6(x = 2a + 3b) = \Psi_7(x = 2a + 3b)$$

$$\therefore,\ C_6^+ \cos k_2(2a+3b) + C_6^- \sin k_2(2a+3b) = Te^{ik_1 b}$$

$$BC\,2:\ j_{6x}\Big|_{x=(2a+3b)} = j_{7x}\Big|_{x=(2a+3b)}$$

$$j_{6x} = \frac{\hbar}{2mi}\left(\Psi_6^* \vec{\nabla}\Psi_6 - \Psi_6 \vec{\nabla}\Psi_6^*\right) = -\frac{k_2}{m} C_6^+ \sin k_2 x + \frac{k_2}{m} C_6^- \cos k_2 x$$

$$j_{7x} = \frac{\hbar}{2mi}\left(\Psi_7^* \vec{\nabla}\Psi_7 - \Psi_7 \vec{\nabla}\Psi_7^*\right) = ik_1 Te^{ik_1 x}$$

$$\therefore,\ \frac{k_2}{m} C_6^+ \sin k_2(2a+3b) + \frac{k_2}{m} C_6^- \cos k_2(2a+3b) = ik_1 Te^{ik_1 b}$$

Again, these equations can be represented in matrix form,

$$\begin{bmatrix} \cosh k_2(2a+3b) & \sinh k_2(2a+3b) \\ -\frac{k_2}{m}\sinh k_2(2a+3b) & \frac{k_2}{m}\cosh k_2(2a+3b) \end{bmatrix} \begin{bmatrix} C_6^+ \\ C_6^- \end{bmatrix} = \begin{bmatrix} Te^{ik_1 b} & 0 \\ ik_1 Te^{ik_1 b} & 0 \end{bmatrix} \begin{bmatrix} T \\ 0 \end{bmatrix}$$

(CONTD……)

Multiple Quantum Wells – Boundary Conditions (VI-VII interface) (E>V_0)

(CONTD.)

$$\begin{bmatrix} \cosh k_2(2a+3b) & \sin k_2(2a+3b) \\ -\dfrac{k_2}{m}\sin k_2(2a+3b) & \dfrac{k_2}{m}\cos k_2(2a+3b) \end{bmatrix} \begin{bmatrix} C_6^+ \\ C_6^- \end{bmatrix} = \begin{bmatrix} Te^{ik_1 b} & 0 \\ ik_1 Te^{ik_1 b} & 0 \end{bmatrix} \begin{bmatrix} T \\ 0 \end{bmatrix}$$

$$\textit{Solving for} \begin{bmatrix} C_6^+ \\ C_6^- \end{bmatrix},$$

$$\begin{bmatrix} C_6^+ \\ C_6^- \end{bmatrix} = \frac{m}{k_2} \begin{bmatrix} \dfrac{k_2}{m}\cos k_2(2a+3b) & -\sin k_2(2a+3b) \\ \dfrac{k_2}{m}\sin k_2(2a+3b) & \cos k_2(2a+3b) \end{bmatrix} \begin{bmatrix} e^{ik_1 b} & 0 \\ ik_1 e^{ik_1 b} & 0 \end{bmatrix} \begin{bmatrix} T \\ 0 \end{bmatrix}$$

x

$$\begin{bmatrix} C_6^+ \\ C_6^- \end{bmatrix} = \left[Y_{VI-VII}\right] \begin{bmatrix} T \\ 0 \end{bmatrix}$$

$$\textit{Subsituting the above in Eq.}16,$$

$$\begin{bmatrix} 1 \\ R \end{bmatrix} = \left[Y_{I-II}\right]\left[Y_{II-III}\right]\left[Y_{III-IV}\right]\left[Y_{IV-V}\right]\left[Y_{V-VI}\right] \begin{bmatrix} C_6^+ \\ C_6^- \end{bmatrix} = \left[Y_{I-II}\right]\left[Y_{II-III}\right]\left[Y_{III-IV}\right]\left[Y_{IV-V}\right]\left[Y_{V-VI}\right]\left[Y_{VI-VII}\right] \begin{bmatrix} T \\ 0 \end{bmatrix} \qquad (Eq.\,17)$$

Multiplying the transfer matrices is again very complex as will be seen from the next slide.

(CONTD…….)

Multiple Quantum Wells –Final Matrix (E>V_0)

The complexity in multiplying these transfer matrices can be seen from the final matrix below:

$$\begin{bmatrix}1\\R\end{bmatrix}=\left[Y_{I-II}\right]\left[Y_{II-III}\right]\left[Y_{III-IV}\right]\left[Y_{IV-V}\right]\left[Y_{V-VI}\right]\left[Y_{VI-VII}\right]\begin{bmatrix}T\\0\end{bmatrix}\quad (Eq.17)$$

$$\begin{bmatrix}1\\R\end{bmatrix}=\underbrace{\begin{bmatrix}\frac{1}{2} & \frac{m}{2ik_1}\\ \frac{1}{2} & \frac{-m}{2ik_1}\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & \frac{k_2}{m}\end{bmatrix}}_{\mathbf{Y_{I\text{-}II}}}$$

$$\times\underbrace{\frac{m}{k_2}\begin{bmatrix}\frac{k_2}{m}\cos k_2 b & -\sin k_2 b\\ \frac{k_2}{m}\sin k_2 b & \cos k_2 b\end{bmatrix}\begin{bmatrix}\cos k_1 b & \sin k_1 b\\ -\frac{k_1}{m}\sin k_1 b & \frac{k_1}{m}\cos k_1 b\end{bmatrix}}_{\mathbf{Y_{II\text{-}III}}}$$

$$\times\underbrace{\frac{m}{k_1}\begin{bmatrix}\frac{k_1}{m}\cos k_1(a+b) & -\sin k_1(a+b)\\ \frac{k_1}{m}\sinh k_1(a+b) & \cos k_1(a+b)\end{bmatrix}\begin{bmatrix}\cos k_2(a+b) & \sin k_2(a+b)\\ -\frac{k_2}{m}\sin k_2(a+b) & \frac{k_2}{m}\cos k_2(a+b)\end{bmatrix}}_{\mathbf{Y_{III\text{-}IV}}}$$

$$\times\underbrace{\frac{m}{k_2}\begin{bmatrix}\frac{k_2}{m}\cos k_2(a+2b) & -\sin k_2(a+2b)\\ \frac{k_2}{m}\sinh k_2(a+2b) & \cos k_2(a+2b)\end{bmatrix}\begin{bmatrix}\cos k_1(a+2b) & \sin k_1(a+2b)\\ -\frac{k_1}{m}\sin k_1(a+2b) & \frac{k_1}{m}\cos k_1(a+2b)\end{bmatrix}}_{\mathbf{Y_{IV\text{-}V}}}$$

$$\times\underbrace{\frac{m}{k_1}\begin{bmatrix}\frac{k_1}{m}\cos k_1(2a+2b) & -\sin k_1(2a+2b)\\ \frac{k_1}{m}\sin k_1(2a+2b) & \cos k_1(2a+2b)\end{bmatrix}\begin{bmatrix}\cos k_2(2a+2b) & \sin k_2(2a+2b)\\ -\frac{k_2}{m}\sin k_2(2a+2b) & \frac{k_2}{m}\cos k_2(2a+2b)\end{bmatrix}}_{\mathbf{Y_{V\text{-}VI}}}$$

$$\times\underbrace{\frac{m}{k_2}\begin{bmatrix}\frac{k_2}{m}\cos k_2(2a+3b) & -\sin k_2(2a+3b)\\ \frac{k_2}{m}\sin k_2(2a+3b) & \cos k_2(2a+3b)\end{bmatrix}\begin{bmatrix}e^{ik_1 b} & 0\\ ik_1 e^{ik_1 b} & 0\end{bmatrix}}_{\mathbf{Y_{VI\text{-}VII}}}\begin{bmatrix}T\\0\end{bmatrix}$$

Multiple Quantum Wells – Reflection and Transmission Probabilities ($E>V_0$)

(CONTD.)

$$\begin{bmatrix} 1 \\ R \end{bmatrix} = \left[Y_{I-II}\right]\left[Y_{II-III}\right]\left[Y_{III-IV}\right]\left[Y_{IV-V}\right]\left[Y_{V-VI}\right]\left[Y_{VI-VII}\right]\begin{bmatrix} \tau \\ 0 \end{bmatrix} \quad (Eq.17)$$

Equation 17 can now be used to calculate the transmission and reflection coefficients. Again, we would need a powerful math tool in order to multiply the complicated transfer matrices. After the transfer matrices are multiplied, we would obtain a 2x2 matrix as shown below:

$$\boxed{\begin{bmatrix} 1 \\ R \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} \tau \\ 0 \end{bmatrix}} \quad (Eq.18)$$

The ***Transmission Co-efficient, "T",*** can then be found by using:

$$\boxed{1 = AT \Rightarrow T = \frac{1}{A}}$$

The ***Reflection Co-efficient, "R",*** would then be given by:

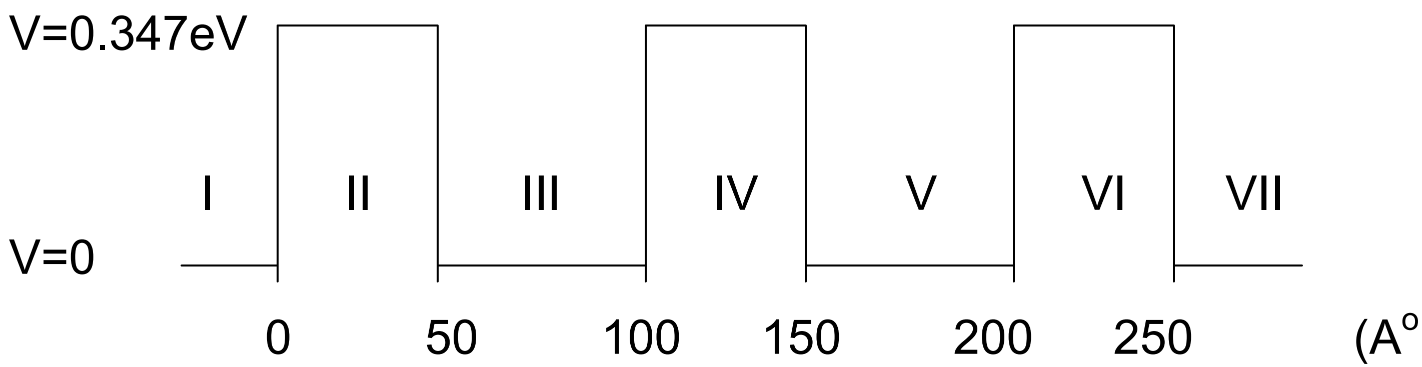
$$\boxed{R = CT = \frac{C}{A}}$$

Therefore, the ***Transmission Probability and Reflection Probability*** is given by:

$$\boxed{TT^* = \left(\frac{1}{A}\right)\left(\frac{1}{A}\right)^* \quad RR^* = 1 - TT^* = 1 - \left(\frac{1}{A}\right)\left(\frac{1}{A}\right)^* \quad OR \quad RR^* = \left(\frac{C}{A}\right)\left(\frac{C}{A}\right)^*}$$

Multiple Quantum Wells – Example 1

Consider a GaAs/AlGaAs multiple quantum well structure with a = 50 A^o and b= 50 A^o:



Multiple Quantum Wells –
Example 1

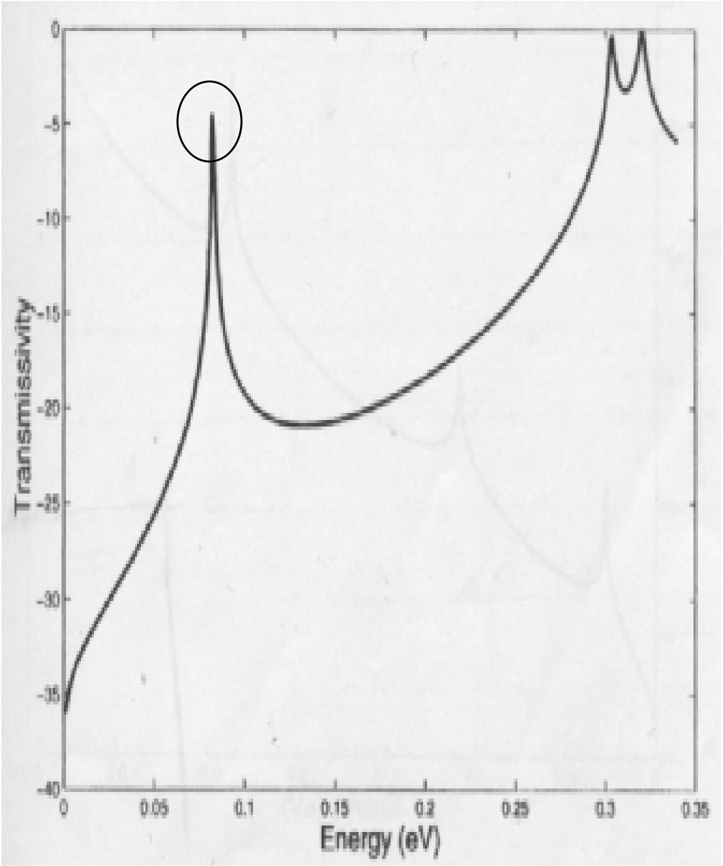
Important Observations from the Transmissivity plot:

- *There are three peak values:*

The peak value of 0.08 eV is the lowest value for which an electron can transmit through the entire structure. The other peak (transmission) values are all above the potential (0.347eV)

- *Transmission is strongly dependent on incident energy* – higher the incident energy, higher the chances of transmission.

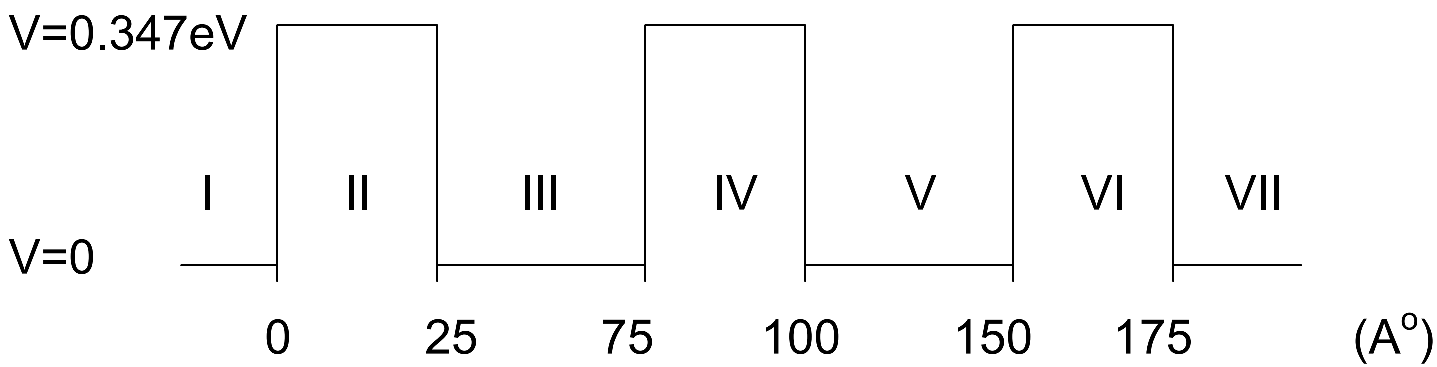
- *The peak values also corresponds to the eigenvalues of the quantum well*. Therefore, its possible to obtain eigenvalues from a transmissivity plot.



Brennan Fig. 2.5.2

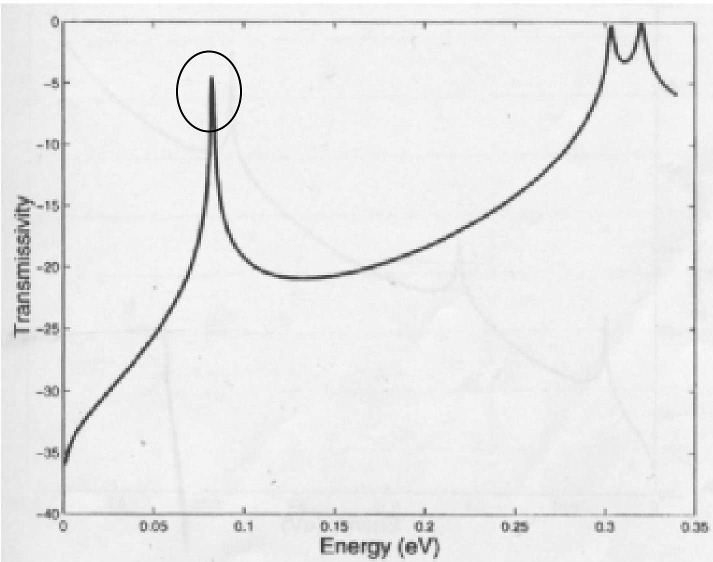
Multiple Quantum Wells – Example 2

Now, consider a GaAs/AlGaAs multiple quantum well structure with a smaller barrier width of 25 A^o:



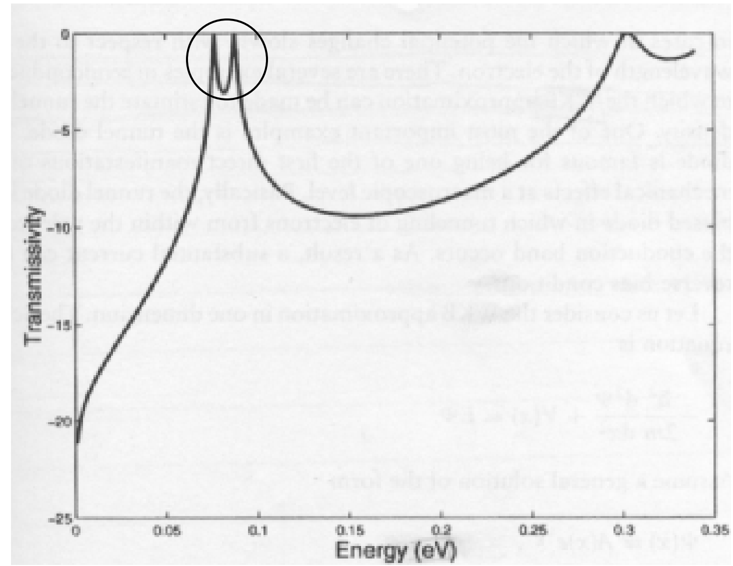
Multiple Quantum Wells – Example 1 versus Example 2

For example 1,



Brennan Fig. 2.5.2

For example 2,



Brennan Fig. 2.5.7

For the transmissivity plot of example 2, there are two peaks at low energy when compared to one peak for the previous example. This is due to the fact that a smaller barrier width was used. Therefore, transmissivity increases with smaller barrier widths.

Multiple Quantum Wells – Additional Comments

- Coupled Quantum Well (Super Lattices)
  - Multiple quantum wells with a non-zero transmission co-efficient throughout the structure
  - More to be discussed in ECE-6453